Craig Interpolation for PDL and its History.

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Applied Logic Seminar, TU Delft
November 24th, 2014

Presenting a proof by Daniel Leivant.
Joint work with Yde Venema.

Outline

The Question

Propositional Dynamic Logic

Definition: Syntax

Atomic propositions p, atomic programs a.

- $\phi := p \mid \neg\phi \mid \phi \vee \phi \mid \phi \wedge \phi \mid \phi \to \phi \mid \langle\alpha\rangle\phi \mid [\alpha]\phi$
- $\alpha := a \mid \alpha;\alpha \mid \alpha \cup \alpha \mid \alpha^* \mid \phi? \mid 1 \mid 0$

Definition: Models

A PDL-model $\mathcal{M} = (W, \mathcal{R}, V)$ consists of

- W: set of worlds/states
- $\mathcal{R} = (R_\xi)_\xi$: family of binary relations on W such that
  - $R_{\chi;\xi} = R_\chi; R_\xi$ (consecution)
  - $R_{\chi\cup\xi} = R_\chi \cup R_\xi$ (union)
  - $R_{\chi^*} = (R_\chi)^*$ (reflexive-transitive closure)
  - $R_{\phi?} = \{(w, w) \in W \times W \mid \mathcal{M}, w \vDash \phi\}$
  - $R_1 = \{(s, t) \in W \times W \mid s = t\}$ (identity on W)
  - $R_0 = \varnothing$ (empty relation)
- $V : \text{Prop} \to \mathcal{P}(W)$: valuation function

Propositional Dynamic Logic

Definition: Truth

- $\mathcal{M}, w \vDash p$ iff $w \in V(p)$
- $\mathcal{M}, w \vDash \neg\phi$ iff $\mathcal{M}, w \nvDash \phi$
- $\mathcal{M}, w \vDash \phi \vee \psi$ iff $\mathcal{M}, w \vDash \phi$ or $\mathcal{M}, w \vDash \psi$
- $\mathcal{M}, w \vDash \phi \wedge \psi$ iff $\mathcal{M}, w \vDash \phi$ and $\mathcal{M}, w \vDash \psi$
- $\mathcal{M}, w \vDash \phi \rightarrow \psi$ iff $\mathcal{M}, w \nvDash \phi$ or $\mathcal{M}, w \vDash \psi$
- $\mathcal{M}, w \vDash \langle\alpha\rangle\phi$ iff there is a $w' \in W$: $wR_\alpha w'$ and $\mathcal{M}, w' \vDash \phi$.
- $\mathcal{M}, w \vDash [\alpha]\phi$ iff for all $w' \in W$: $wR_\alpha w'$ also $\mathcal{M}, w' \vDash \phi$.

Definition: Validity

A formula ϕ is *valid* iff it is true at all states in all models.
In this case we write $\vDash \phi$.

Craig Interpolation

Definition: Language
The language of a formula ϕ is the set $L(\phi)$ consisting of all atomic propositions and programs occurring in ϕ.

Example: $L([a;b]p \to \langle c \rangle q) = \{a, b, c, p, q\}$

Definition: Interpolation
A logic has *Craig Interpolation* iff for all formulas ϕ and ψ such that $\models \phi \to \psi$ there is a formula θ called *interpolant* such that

- $\models \phi \to \theta$
- $\models \theta \to \psi$
- $L(\theta) \subseteq L(\phi) \cap L(\psi)$

Example: q is an interpolant for $\models (p \wedge q) \to (q \vee r)$.

Propositional logic, first-order logic, intuitionistic logic, basic and multi-modal logic and the μ-calculus have Craig Interpolation.

Hall of Fame and Failure

Known proof attempts:

1. Daniel Leivant: *Proof theoretic methodology for propositional dynamic logic*. LNCS, 1981.
2. Manfred Borzechowski: *Tableau–Kalkül für PDL und Interpolation*. Diploma thesis, FU Berlin, 1988. Unpublished.
3. Tomasz Kowalski: *PDL has interpolation*. JSL, 2002. Revoked in 2004.

Hall of Fame and Failure

Other notable references:

1. Marcus Kracht: Chapter *The open question* in *Tools and techniques in modal logic*, 1999.
2. D'Agostino & Hollenberg: *Logical questions concerning the μ-calculus: interpolation, Lyndon and Łoś-Tarski*. JSL, 2000.
3. Johan van Benthem: *The many faces of Interpolation*. Synthese, 2008.

Honesty is a Virtue

THE JOURNAL OF SYMBOLIC LOGIC
Volume 69, Number 3, Sept. 2004

RETRACTION NOTE FOR
"**PDL** HAS INTERPOLATION"

TOMASZ KOWALSKI

In this journal I published a paper [1] entitled "**PDL** has interpolation" purporting to prove what the title announced. It has been pointed out to me by Yde Venema that my argument contains a serious error. As I have not been able to correct it, the problem of interpolation for Propositional Dynamic Logic is still open.

Kracht VS. Leivant

"***Chapter 10.6: The Unanswered Question***
[...] the problem of interpolation for PDL. This is one of the major open problems in this area. Twice a solution has been announced, in [Leivant 1981] and [Borzechowski 1988], but in neither case was it possible to verify the argument."

Marcus Kracht: *Tools and techniques in modal logic* (1999)

It's a mess.

(It is an open question whether)[2] PDL has Craig-Interpolation.

Leivant 1981, revised

Leivant 1981

Proof theoretic methodology for Propositional Dynamic Logic

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Abstract. We relate by syntactic techniques finitary and infinitary axiomatizations for the iterator-construct * of Propositional Dynamic Logic PDL. This is applied to derive the Interpolation Theorem for PDL, and to provide a new proof of the semantic completeness of Segerberg's axiomatic system for PDL.

Contrary to semantic techniques used to date, our proof of completeness is relatively insensitive to changes in the language and axioms used, provided some minimum syntactic closure properties hold. For instance, the presence of the test-operator adds no difficulty, and the proof also establishes the Interpolation Theorem and the closure under iteration of a constructive variant of PDL.

Simplifying the question

Completeness of Segerberg's axioms is also shown by Leivant, but not our interest here.

In 2014 we know:

- PDL does not have uniform interpolation. [1]
- Test-free PDL has interpolation iff PDL has. [8]

Hence, reduce the syntax to:

- $\phi := p \mid \neg\phi \mid \phi \to \phi \mid [\alpha]\phi$
- $\alpha := a \mid \alpha;\alpha \mid \alpha \cup \alpha \mid \alpha^* \mid 1 \mid 0$

Ignore tests.
Let $\vee$, $\wedge$ and $\langle\alpha\rangle$ be the appropriate abreviations.

Steps of the proof

1. Define a sound and complete sequent calculus for PDL.
2. Use Maehara's method to show Partition-Interpolation.
   - 2.1 Show that the calculus has the "step-by-step property".
   - 2.2 For the $*$ case, find a repetitive scheme in long enough proofs.
   - 2.3 Use linear transformations of programs to imply a $*$ formula.
3. Check that Partition-Interpolation implies Craig Interpolation.

A sequent calculus for PDL

Notation

X, Y, Z: formulas

f, g: sets of formulas

α, β: programs

Sequent example: $f, X \vdash \phi$

Proof example

$$\dfrac{\dfrac{\dfrac{\dfrac{[a]p \vdash [a]p}{[a]p, [b]p \vdash [b]p}\,\text{WEAK} \qquad \dfrac{[b]p \vdash [b]p}{[a]p, [b]p \vdash [b]p}\,\text{WEAK}}{[a]p, [b]p \vdash [a \cup b]p}\,(\cup\text{R})}{[a] \vdash [b]p \to [a \cup b]p}\,(\to \text{R})}{\vdash [a]p \to ([b]p \to [a \cup b]p)}\,(\to \text{R})$$

A sequent calculus for PDL

Let CD be the following proof system where g is $\varnothing$ or a singleton.

$$(\neg\mathrm{R})\ \frac{f, X \vdash}{f, \vdash \neg X} \qquad (\neg\mathrm{L})\ \frac{f \vdash X}{f, \neg X \vdash}$$

$$(\rightarrow \mathrm{R})\ \frac{f, X \vdash Y}{f \vdash X \rightarrow Y} \qquad (\rightarrow \mathrm{L})\ \frac{f \vdash X \qquad f, Y \vdash g}{f, X \rightarrow Y \vdash g}$$

$$(;\mathrm{R})\ \frac{f \vdash [\alpha][\beta]X}{f \vdash [\alpha;\beta]X} \qquad (;\mathrm{L})\ \frac{f, [\alpha][\beta]X \vdash g}{f, [\alpha;\beta]X \vdash g}$$

$$(\cup\mathrm{R})\ \frac{f \vdash [\alpha]X \qquad f \vdash [\beta]X}{f \vdash [\alpha \cup \beta]X} \qquad (\cup\mathrm{L})\ \frac{f, [\alpha]X, [\beta]X \vdash g}{f, [\alpha \cup \beta]X \vdash g}$$

$$(*L)\ \frac{f, X, [\alpha][\alpha^*]X \vdash g}{f, [\alpha^*]X \vdash g} \qquad (*\mathrm{R})\ \frac{f \vdash \phi \qquad f \vdash [\alpha]\phi \qquad \cdots \qquad f \vdash [\alpha]^k\phi}{f \vdash [\alpha^*]\phi}$$

where $k = 2^{|f|+|\phi|}$

$$(\mathrm{GEN})\ \frac{f \vdash X}{[\alpha]f \vdash [\alpha]X} \qquad (\mathrm{WEAK})\ \frac{f \vdash g}{f' \vdash g'}$$

where $f \subseteq f'$ and $g \subseteq g'$.

A sequent calculus for PDL

Theorem (Leivant 1981)
CD is a intuitionistic/constructive variant of D which is a sound an complete system for PDL, i.e. we have:

$$\vDash X \text{ iff } \vdash_{\mathrm{D}} X \text{ iff } \vdash_{\mathrm{CD}} X^0$$

where X^0 is the result of inserting $\neg\neg$ in front of everything in X.

NB: CD is *not* sound and complete for intuitionistic/constructive PDL.

Remaining goal: Show that CD has interpolation.

Maehara's method

Idea

Find interpolants by going along the proof tree.
Given the previous interpolants, we define the next one.

Example

Suppose the last step is $\cup$R:

$$\dfrac{\dfrac{\vdots}{f \vdash [\alpha]X} \qquad \dfrac{\vdots}{f \vdash [\beta]X}}{f \vdash [\alpha \cup \beta]X}\,(\cup\mathrm{R})$$

Given any two interpolants Z_1 and Z_2 for $f \vdash [\alpha]X$ and $f \vdash [\beta]X$, let $Z := Z_1 \wedge Z_2 = \neg(Z_1 \to \neg Z_2)$. This interpolates $f \vdash [\alpha \cup \beta]X$.

Partition-Interpolation

Definition
Given a sequent $f \vdash X$ and a partition of f into $f^-; f^+$, we say that K is an interpolant for $f^-; f^+ \vdash X$ iff

$$L(K) \subseteq L(f^-) \cap L(f^+, X) \quad \text{and} \quad f^- \vdash K \quad \text{and} \quad f^+, K \vdash X$$

Lemma 5.3.1 (Leivant 1981)
Let $f^-; f^+$ be any partition of f and q not occur in f.

(i) If $f \vdash_{\mathsf{CD}} X$, then there is an interpolant for $f^-; f^+ \vdash X$.

(ii) Suppose P is a proof of $f \vdash [\alpha]q$ from $\{f_i \vdash q\}_{i<k}$ and let $f_i^-; f_i^+$ be the partitions of f_i induced by $f^-; f^+$ for all $i < k$. If K_i is an interpolant for $f_i^-; f_i^+ \vdash X$ for all $i < k$, then there is an interpolant of the form $\bigwedge_i [\beta_i] K_i$ for $f^-; f^+ \vdash [\alpha]X$.

Proof. By tree-induction on P, simultaneously for (i) and (ii).

Partition-Interpolation

An easy warm-up case

Suppose the last step is $\to$ L:

$$\frac{f \vdash X \qquad f, Y \vdash Z}{f, X \to Y \vdash Z} (\to \mathrm{L})$$

Case a) partition $f^-, X \to Y; f^+$. By induction hypothesis:

- $f^+; f^- \vdash X$ (Note: flipped!) yields K_1 such that

  $$L(K_1) \subseteq L(f^+) \cap L(f^-, X) \text{ and } f^+ \vdash K_1 \text{ and } f^-, K_1 \vdash X$$

- $f^-, Y; f^+ \vdash Z$ yields K_2 such that

  $$L(K_2) \subseteq L(f^-, Y) \cap L(f^+, Z) \text{ and } f^-, Y \vdash K_2 \text{ and } f^+, K_2 \vdash Z$$

Let $K := K_1 \to K_2$. This is interpolates $f^-, X \to Y; f^+ \vdash Z$.

Case b) partition $f^-; X \to Y, f^+$. Then $K := K_1 \wedge K_2$ works.

Partition-Interpolation

The evil $*$ case

Suppose the last step of P is $(*\mathrm{R})$. For each $h = 1 \leq M$ let P_h be the proof of $f \vdash [\alpha]^h X$ occurring in P above this premise:

$$\dfrac{\dfrac{P_0}{f \vdash X} \quad \dfrac{P_1}{f \vdash [\alpha]X} \quad \cdots \quad \dfrac{P_M}{f \vdash [\alpha]^M X}}{f \vdash [\alpha^*]X}\,(*\mathrm{R})$$

Note: all active formulas on the right. Hence, only consider the given partition f^-, f^+ without further manipulation.

Given: M many interpolants. Goal: find a formula K such that

$$L(K) \subseteq L(f^-) \cap L(f^+, [\alpha^*]X) \text{ and } f^- \vdash K \text{ and } f^+, K \vdash [\alpha^*]X$$

How?!

Positive Closure

Definition

The *positive closure* of f, denoted by $\mathrm{PC}(f)$, is the smallest set $g \supseteq f$ such that:

- If $(X \to Y) \in g$, then $Y \in g$.
- If $[\alpha]X \in g$, then $X \in g$.
- If $[\alpha;\beta]X \in g$, then $[\alpha][\beta]X \in g$.
- If $[\alpha \cup \beta]X \in g$, then $[\alpha]X \in g$ and $[\beta]X \in g$.
- If $[\alpha^*]X \in g$, then $[\alpha][\alpha^*]X \in g$.

Note: Whenever f is finite, $\mathrm{PC}(f)$ is also finite.

Nice property 1

In certain proofs, PC($\cdot$) is preserved in the following sense.

Lemma 4.2.1 (Leivant 1981, revision Venema 2014)
If P proves $f \vdash [\beta_1] \dots [\beta_k][\alpha]^m q$ from $\{f_i \vdash q\}_i$ where $q \notin L(f)$, all β_is are subprograms of α, $r < m$ and $f' \vdash [\alpha]^r q$ is a sequent in P (under a non-initial leaf) then $PC(f') \subseteq PC(f)$.

The case we need is $k = 0$.

Nice property 2

Definition
Let $P[X/q]$ be the result of substituting X for q in P.

Lemma 4.2.2 (Leivant 1981)
Suppose P proves $f \vdash [\alpha]^r X$ from $\{f_i \vdash X\}_i$ where $X \notin PC(f)$. Then there is a proof P' of $f \vdash [\alpha]^r q$ from $\{f_i' \vdash q\}_i$ such that $P = P'[X/q]$.

Intuitively, this means that P does not take X apart:

$$\begin{array}{c} \{f_i \vdash X\}_i \\ \hline \vdots \\ \hline f \vdash [\alpha]^r X \end{array} = \left(\begin{array}{c} \{f_i \vdash q\}_i \\ \hline \vdots \\ \hline f \vdash [\alpha]^r q \end{array} \right) [X/q]$$

Nice property 3: Step by Step

Suppose P is a CD-proof of $f \vdash [\alpha]^n X$. Then P consists of proof parts $P_0, \ldots, P_n$ which build up the $[\alpha]$s "step by step":

$$
\begin{array}{c}
P_0 \\ \hline
\{f_j \vdash X\}_{j \in I_0} \\ \hline
P_1 \\ \hline
\{f_j \vdash [\alpha]X\}_{j \in I_1} \\ \hline
P_2 \\ \hline
\{f_j \vdash [\alpha]^2 X\}_{j \in I_2} \\ \hline
\vdots \\ \hline
\{f_j \vdash [\alpha]^{n-1} X\}_{j \in I_{n-1}} \\ \hline
P_n \\ \hline
f \ \vdash \ [\alpha]^n X
\end{array}
$$

NB: This looks more linear than it actually is!

Linear Transformations

Think of programs and formulas as a vector space:

$$(\beta)\vec{Y} = \begin{pmatrix} \beta_{1,1} & \cdots & \beta_{1,k} \\ \vdots & \ddots & \vdots \\ \beta_{k,1} & \cdots & \beta_{k,k} \end{pmatrix} \begin{pmatrix} Y_1 \\ \vdots \\ Y_k \end{pmatrix} := \begin{pmatrix} [\beta_{1,1}]Y_1 \wedge \cdots \wedge [\beta_{1,k}]Y_k \\ \vdots \\ [\beta_{k,1}]Y_1 \wedge \cdots \wedge [\beta_{k,k}]Y_k \end{pmatrix}$$

Lemma

For every $k \times k$ matrix (β) there exists a (γ) such that

$$(\gamma) \equiv (\beta)^* = (\beta)(\beta)(\beta)\ldots$$

Linear Transformations

Example

Let $\vec{Y} = \langle p, q \rangle$ and $(\beta) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $(\beta)\vec{Y} = \begin{pmatrix} [a]p \wedge [b]q \\ [c]p \wedge [d]q \end{pmatrix}$

and $(\beta)(\beta)\vec{Y} = \begin{pmatrix} [a]([a]p \wedge [b]q) \wedge [b]([c]p \wedge [d]q) \\ [c]([a]p \wedge [b]q) \wedge [d]([c]q \wedge [d]q) \end{pmatrix} \ldots$

Let $\gamma := \begin{pmatrix} (a \cup (b; (d^*; c)))^* & (a^*; b)((c; a^*; b) \cup d)^* \\ (d^*; c)(a \cup (b; (d^*; c)))^* & ((c; a^*; b) \cup d)^* \end{pmatrix}$

Then $(\gamma) \equiv (\beta)^*$ and $(\beta)^*\vec{Y} \equiv (\gamma)\vec{Y}$.

This γ can be found systematically. Moreover, it is useful:

$$p \wedge [a]p \wedge ([a]p \wedge [b]q) \wedge ([a]([a]p \wedge [b]q) \wedge [b]([c]p \wedge [d]q)) \wedge \ldots$$

$$\equiv \quad [(a \cup (b; (d^*; c)))^*]p \wedge [(a^*; b)((c; a^*; b) \cup d)^*]q$$

Putting it all together

Back to the evil $*$ case

Now we can deal with this:

$$\dfrac{\dfrac{P_0}{f \vdash X} \quad \dfrac{P_1}{f \vdash [\alpha]X} \quad \cdots \quad \dfrac{P_M}{f \vdash [\alpha]^M X}}{f \vdash [\alpha^*]X}\ (*\mathrm{R})$$

Fix a ridiculously large $h := s + v + d$ where

- d such that $[\alpha]^d X \notin \mathrm{PC}(f)$
- $v := 2^{|\mathrm{PC}(f)|} \cdot 2^{|f|} + 1$
- $s := 1$ (for now).

Apply the step by step property to P_h:

$$\dfrac{\dfrac{\dfrac{Q_i}{\{f_i^-; f_i^+ \vdash [\alpha]^d X\}_{i \in I_d}}}{\vdots}}{f^-; f^+ \vdash [\alpha]^{d+v+s} X}$$

Putting it all together

Finding a repetitive pattern

Now P_h has to look like this:

$$\dfrac{\dfrac{\dfrac{\dfrac{Q_{j,i}}{\{f_i^-; f_i^+ \vdash [\alpha]^d X\}_{i \in I_d}}}{R'_j[[\alpha]^d X/q]}}{\{f_j^-; f_j^+ \vdash [\alpha]^{d+v} X\}_{j \in I_{d+v}}}}{\dfrac{U'[[\alpha]^{d+v} X/q]}{f^-; f^+ \vdash [\alpha]^{d+v+s} X}}$$

For all $c \leq v$, $j \in I_{d+c}$: $f_i \subseteq \mathrm{PC}(f_i) \subseteq \mathrm{PC}(f)$ and $|\mathcal{P}(f_j)| \leq |\mathcal{P}(f)|$

Hence $| \cup \{\mathcal{P}(f_j) \mid c \leq v, j \in I_{d+c}\} \leq |\mathcal{P}(\mathrm{PC}(f)| \cdot |\mathcal{P}(f)|$
$= 2^{|\mathrm{PC}(f)|} \cdot 2^{|f|} = v - 1 < v$.

Repetitive Pattern

For some $m \neq n$ we have $\{f_j^+; f_j^- \mid j \in I_m\} = \{f_j^+; f_j^- \mid j \in I_n\}$.
Furthermore, we can assume $d < m < n < d + v$ and $I_m = I_n$.

Putting it all together

Applying the induction hypothesis

Let r be such that $n = m + r$. Now P_h can be divided as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{Q_i}{\{f_i^-; f_i^+ \vdash [\alpha]^m X\}_{i \in I}}}{R'_j[[\alpha]^m X/q]}}{\{f_j^-; f_j^+ \vdash [\alpha]^{m+r} X\}_{j \in I}}}{\cfrac{U'[[\alpha]^{m+r} X/q]}{f_i^-; f_i^+ \vdash [\alpha]^{m+r+s} X}}$$

IH(i) yields $\vec{K}$ such that K_i interpolates $f_i^-; f_i^+ \vdash [\alpha]^m X$. Using IH(ii) r times: If $\vec{M}$ contains interpolants for $f_i^-; f_i^+ \vdash Y$ then there is a matrix (β) such that $((\beta)M)_i$ interpolates $f_i^-; f_i^+ \vdash [\alpha]^r Y$.

Thus, for all n, by applying the latter to the former n times:

$$f_i^- \vdash ((\beta)^n K)_i \quad \text{and} \quad f_i^+, ((\beta)^n K)_i \vdash [\alpha]^m [\alpha]^{r \times n} X$$

Putting it all together

Done, repeat.

By linear transformations there is a γ such that:

$$f_i^- \vdash ((\gamma)K)_i \quad \text{and} \quad f_i^+, ((\gamma)K)_i \vdash [\alpha]^m[(\alpha^r)^*]X$$

Now apply IH(ii) to all the $((\gamma)K)_i$s and U'.
This yields an interpolant H_s for $f^-; f^+ \vdash [\alpha]^s[\alpha]^m[(\alpha^r)^*]X$.
Repeat all of the above to obtain $H_1, \ldots, H_{v+d}$.

Finally, let $K := \bigwedge_{s \leq v+d} H_s$. This interpolates $f^-; f^+ \vdash [\alpha^*]X$.

Lemma
$\vdash_{\mathrm{CD}} \bigwedge_{k<w} [\alpha^k][(\alpha^w)^*]X \to [\alpha^*]X$.

Putting it all together

This is the end.

Theorem 5.3.2 (i) (Leivant 1981)
PDL has Craig Interpolation.

Proof. Take any $\vDash X \to Y$. D is complete, hence $\vdash_D X \to Y$.
Then $\vdash_{CD} X^o \to Y^o$ and thus $X^o \vdash_{CD} Y^o$.
Partition-interpolation of $X^o; \varnothing \vdash Y^o$ yields Z such that

- $L(Z) \subseteq L(X^o) \cap L(\varnothing, Y^o)$,
- $X^o \to Z \in$ **PDL** and $Z \to Y^o \in$ **PDL**

By $X^o \equiv X$, $Y^o \equiv Y$, $L(X^o) = L(X)$ and $L(Y^o) = L(Y)$:

- $L(Z) \subseteq L(X) \cap L(Y)$,
- $X \to Z \in$ **PDL** and $Z \to Y \in$ **PDL**

Hence Z is an interpolant for $X \to Y$.

$\square$

Criticism

Criticism

Marcus Kracht: *Tools and techniques in modal logic*. (1999)
Chapter *10.6. The Unanswered Question*:

> *"[T]he problem of interpolation for* ***PDL*** *is one of the major open problems in this area. Twice a solution has been announced [...], but in neither case was it possible to verify the argument.*
> *The argument of Leivant makes use of the fact that if* $\phi \vdash_{\textbf{PDL}} \psi$ *then we can bound the size of a possible countermodel so that the star* α^* *only needs to search up to a depth d which depends on* ϕ *and* ψ*."[8, p. 493]*

Criticism

Marcus Kracht (continued):

> *"The argument of Leivant makes use of the fact that if $\phi \vdash_{\mathbf{PDL}} \psi$ then we can bound the size of a possible countermodel so that the star α^* only needs to search up to a depth d which depends on ϕ and ψ. Once that is done, we have reduced **PDL** to **EPDL**, which definitely has interpolation because it is a notational variant of polymodal **K**. However, this is tantamount to the following. Abbreviate by $\mathbf{PDL}^n$ the strengthening of **PDL** by axioms of the form $[a^*]p \leftrightarrow [a^{\leq n}]p$ for all a. Then, by the finite model property of **PDL**, **PDL** is the intersection of the logics $\mathbf{PDL}^n$. Unfortunately, it is not so that interpolation is preserved under intersection."[8, p. 493]*

PDL and PDL^n

Definition

Semantic closure $\mathsf{SCL}(A) := \{\phi \mid A \vDash \phi\}$

$[\alpha^{\leq n}]\phi := \phi \wedge [\alpha]\phi \wedge [\alpha;\alpha]\phi \wedge \cdots \wedge [\alpha^n]\phi$

$\mathbf{PDL}^n := \mathsf{SCL}\left(\mathbf{PDL} \cup \{[\alpha^*]p \leftrightarrow [\alpha^{\leq n}]p \mid \alpha \in PROG, p \in \mathfrak{P}\}\right)$

Theorem

$$\mathbf{PDL}^0 \supseteq \mathbf{PDL}^1 \supseteq \mathbf{PDL}^2 \supseteq \cdots \supseteq \mathbf{PDL} = \bigcap_n \mathbf{PDL}^n$$

Idea / Question

Is there an n, depending on $|\phi \to \psi|$ such that any $\mathbf{PDL}^n$-interpolant for $\phi \to \psi$ is also a **PDL**-interpolant?

Criticism

But this is not what Leivant is doing:

$$(*\mathrm{R})\ \frac{f \vdash \phi \qquad f \vdash [\alpha]\phi \qquad \cdots \qquad f \vdash [\alpha]^k\phi}{f \vdash [\alpha^*]\phi}$$

where $k = 2^{|f|+|\phi|}$ and therefore depends on f and ϕ.

Theorem: Finite-Model Property
If ϕ is satisfiable, then there is a model $\mathcal{M} = (W, \mathcal{R}, V)$ and a world $w \in W$ such that $\mathcal{M}, w \vDash \phi$ and $|W| \leq 2^{\mathsf{size}(\phi)}$.

Lemma
If $\vDash \bigwedge f \to [\alpha]^n\phi$ for all $n \leq k = 2^{|f|+|\phi|}$, then $\vDash \bigwedge f \to [\alpha^*]\phi$.

Theorem
The finitary rule is admissible.

Conclusion

Conclusion

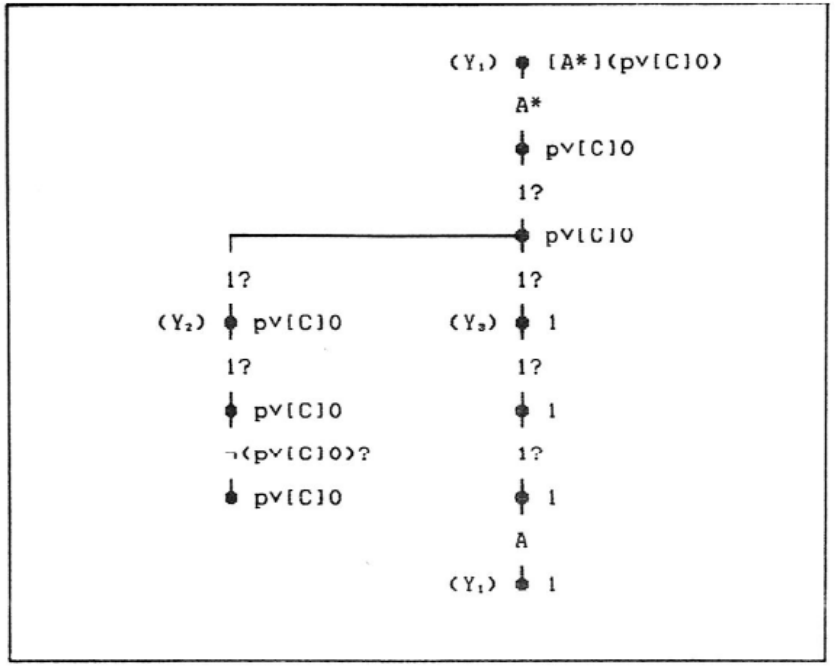
- There is a finitary sequent calculus for **PDL**.
  (In particular, Kracht's criticism does not apply.)
- This system has the "step by step" property.
- Therefore we can:
  - find a repetitive pattern in long enough proofs.
  - use linear transformations to build $*$ interpolants.
- This extends Maehara's method to show Craig Interpolation.

All this [c $\cup$ sh]ould have been known since 1981.

Moreover, can this proof also be done in multi-type calculi?

Epilogue

Kracht: "**Twice** a solution has been announced ..."
Borzechowski 1988: unpublished, unknown and unread?



Ergebnis der Konstruktion ist also, daß I_0=[A*](p∨[C]0) ein Interpoland für die Formeln [(A;A)*](p∧[A;(B∪C)]0) und [A*](p∨[C]q) ist. Auf Grund der vorgenommenen Überlegungen, einen nicht unnötig großen Interpolanden zu konstruieren, ist I_0 sogar frei von Tests. Dieses Ergebnis ist jedoch nicht immer erreichbar.

References I

Giovanna D'Agostino, Marco Hollenberg: *Logical Questions Concerning the μ-Calculus: Interpolation, Lyndon and Łoś-Tarski*. In: *Journal of Symbolic Logic*, Volume 65, Issue 1, March 2000, pp 310-332.
http://www.jstor.org/stable/2586539

Johan van Benthem: *The many faces of interpolation*.
In: *Synthese*, Volume 164, 2008, pp 451–460.
http://link.springer.com/article/10.1007/s11229-008-9351-5

Manfred Borzechowski: *Tableau–Kalkül für PDL und Interpolation*.
Diplomarbeit, Department of Mathematics, FU Berlin, 1988.
Unpublished. Soon available at http://www.borzechowski.de/

Michael J. Fischer, Richard E. Ladner: *Propositional dynamic logic of regular programs*.
In: *Journal of Computer and System Sciences*, Volume 18, Issue 2, April 1979, pp 194-211.
http://dx.doi.org/10.1016/0022-0000(79)90046-1

Amélie P. Gheerbrant: *Fixed-point logics on trees*.
Dissertation, University of Amsterdam 2010.
http://dare.uva.nl/en/record/360229

Tomasz Kowalski: *PDL has Interpolation*.
In: *Journal of Symbolic Logic*, Volume 67, Issue 3, September 2002, pp 933-946.
http://www.jstor.org/stable/3648548

Tomasz Kowalski: *Retraction note for "PDL has interpolation"*.
In: *Journal of Symbolic Logic*, Volume 69, Issue 3, September 2004, p 935.
http://www.jstor.org/stable/30041768

References II

Marcus Kracht: *Tools and techniques in modal logic*.
Series: *Studies in logic and the foundations of mathematics*, Volume 142, Elsevier, 1999.
http://wwwhomes.uni-bielefeld.de/mkracht/html/tools/book.pdf

Dexter Kozen: *A Completeness Theorem for Kleene Algebras and the Algebra of Regular Events*. In: *Information and Computation*, Volume 110, Issue 2, 1994, pp 366–390.
http://dx.doi.org/10.1006/inco.1994.1037

Dexter Kozen: *On Action Algebras*.
In: Jan van Eijck, Albert Visser (Editors): *Logic and Information Flow*. MIT Press, 1994, pp 78-88.
http://books.google.com/books?isbn=0262220474

Daniel Leivant: *Proof theoretic methodology for propositional dynamic logic*.
In: *Lecture Notes in Computer Science*, Volume 107, 1981, pp 356-373.
http://dx.doi.org/10.1007/3-540-10699-5_111

Judit Madarász: *The Craig interpolation theorem in multi-modal logics*.
In: *Bulletin of the Section of Logic*, Volume 3.24, 1995, pp 147-151.
http://www.filozof.uni.lodz.pl/bulletin/pdf/24_3_5.pdf

Krister Segerberg: *A completeness theorem in the modal logic of programs*.
In: *Notices American Mathematical Society*, Volume 24, 1977, A522.
Republished in: *Banach Center Publications*, Volume 9.1, 1982, pp 31-46.
https://eudml.org/doc/209235

Gaisi Takeuti: *Proof Theory*. North-Holland, Amsterdam/Oxford 1975.
http://www.getcited.org/pub/101589483

Thank you!

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