Craig Interpolation for PDL and its History.

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Presenting a proof by Daniel Leivant.  
Joint work with Yde Venema.
Outline

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The Question
Propositional Dynamic Logic

Definition: Syntax

Atomic propositions $p$, atomic programs $a$.

$\phi ::= p \mid \neg \phi \mid \phi \lor \phi \mid \phi \land \phi \mid \phi \rightarrow \phi \mid \langle \alpha \rangle \phi \mid [\alpha] \phi$

$\alpha ::= a \mid \alpha; \alpha \mid \alpha \cup \alpha \mid \alpha^* \mid \phi? \mid 1 \mid 0$

Definition: Models

A PDL-model $\mathcal{M} = (W, \mathcal{R}, V)$ consists of

- $W$: set of worlds/states
- $\mathcal{R} = (R_\xi)_{\xi}$: family of binary relations on $W$ such that
  - $R_{\chi;\xi} = R_{\chi} \cdot R_\xi$ (consecution)
  - $R_{\chi \cup \xi} = R_{\chi} \cup R_\xi$ (union)
  - $R_{\chi^*} = (R_{\chi})^*$ (reflexive-transitive closure)
  - $R_{\phi?} = \{(w, w) \in W \times W \mid \mathcal{M}, w \models \phi\}$
  - $R_1 = \{(s, t) \in W \times W \mid s = t\}$ (identity on $W$)
  - $R_0 = \emptyset$ (empty relation)
- $V : \text{Prop} \to \mathcal{P}(W)$: valuation function
Definition: Truth

- $\mathcal{M}, w \models p$ iff $w \in V(p)$
- $\mathcal{M}, w \models \neg \phi$ iff $\mathcal{M}, w \not\models \phi$
- $\mathcal{M}, w \models \phi \lor \psi$ iff $\mathcal{M}, w \models \phi$ or $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \phi \land \psi$ iff $\mathcal{M}, w \models \phi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \phi \rightarrow \psi$ iff $\mathcal{M}, w \not\models \phi$ or $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \langle \alpha \rangle \phi$ iff there is a $w' \in W$: $wR_{\alpha}w'$ and $\mathcal{M}, w' \models \phi$.
- $\mathcal{M}, w \models [\alpha] \phi$ iff for all $w' \in W$: $wR_{\alpha}w'$ also $\mathcal{M}, w' \models \phi$.

Definition: Validity

A formula $\phi$ is valid iff it is true at all states in all models.
In this case we write $\models \phi$. 
Craig Interpolation

**Definition: Language**
The language of a formula $\phi$ is the set $L(\phi)$ consisting of all atomic propositions and programs occurring in $\phi$.

**Example:** $L([a; b]p \rightarrow ⟨c⟩q) = \{a, b, c, p, q\}$

**Definition: Interpolation**
A logic has *Craig Interpolation* iff for all formulas $\phi$ and $\psi$ such that $\models \phi \rightarrow \psi$ there is a formula $\theta$ called *interpolant* such that

1. $\models \phi \rightarrow \theta$
2. $\models \theta \rightarrow \psi$
3. $L(\theta) \subseteq L(\phi) \cap L(\psi)$

**Example:** $q$ is an interpolant for $\models (p \land q) \rightarrow (q \lor r)$.

Propositional logic, first-order logic, intuitionistic logic, basic and multi-modal logic and the $\mu$-calculus have Craig Interpolation.
Hall of Fame and Failure

Known proof attempts:


Other notable references:


RETRACTION NOTE FOR
“PDL HAS INTERPOLATION”

TOMASZ KOWALSKI

In this journal I published a paper [1] entitled “PDL has interpolation” purporting to prove what the title announced. It has been pointed out to me by Yde Venema that my argument contains a serious error. As I have not been able to correct it, the problem of interpolation for Propositional Dynamic Logic is still open.
“Chapter 10.6: The Unanswered Question

[...] the problem of interpolation for PDL. This is one of the major open problems in this area. Twice a solution has been announced, in [Leivant 1981] and [Borzechowski 1988], but in neither case was it possible to verify the argument.”

Marcus Kracht: Tools and techniques in modal logic (1999)
It’s a mess.

(It is an open question whether)² PDL has Craig-Interpolation.
Leivant 1981, revised
Proof theoretic methodology for Propositional Dynamic Logic

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Abstract. We relate by syntactic techniques finitary and infinitary axiomatizations for the iterator-construct * of Propositional Dynamic Logic PDL. This is applied to derive the Interpolation Theorem for PDL, and to provide a new proof of the semantic completeness of Segerberg's axiomatic system for PDL.

Contrary to semantic techniques used to date, our proof of completeness is relatively insensitive to changes in the language and axioms used, provided some minimum syntactic closure properties hold. For instance, the presence of the test-operator adds no difficulty, and the proof also establishes the Interpolation Theorem and the closure under iteration of a constructive variant of PDL.
Simplifying the question

Completeness of Segerberg’s axioms is also shown by Leivant, but not our interest here.

In 2014 we know:

- PDL does not have uniform interpolation. [1]
- Test-free PDL has interpolation iff PDL has. [8]

Hence, reduce the syntax to:

- \( \phi := p \mid \neg \phi \mid \phi \rightarrow \phi \mid [\alpha] \phi \)
- \( \alpha := a \mid \alpha; \alpha \mid \alpha \cup \alpha \mid \alpha^* \mid 1 \mid 0 \)

Ignore tests.
Let \( \lor, \land \) and \( \langle \alpha \rangle \) be the appropriate abbreviations.
Steps of the proof

1. Define a sound and complete sequent calculus for PDL.

2. Use Maehara's method to show Partition-Interpolation.
   
   2.1 Show that the calculus has the “step-by-step property”.
   
   2.2 For the * case, find a repetitive scheme in long enough proofs.
   
   2.3 Use linear transformations of programs to imply a * formula.

3. Check that Partition-Interpolation implies Craig Interpolation.
A sequent calculus for PDL

**Notation**

$X$, $Y$, $Z$: formulas

$f$, $g$: sets of formulas

$\alpha$, $\beta$: programs

**Sequent example:** $f, X \vdash \phi$

**Proof example**

\[
\frac{[a]p \vdash [a]p}{[a]p, [b]p \vdash [b]p} \text{ WEAK} \quad \frac{[b]p \vdash [b]p}{[a]p, [b]p \vdash [b]p} \text{ WEAK} \quad \frac{[b]p \vdash [b]p}{[a]p, [b]p \vdash [b]p} \text{ (UR)}
\]

\[
\frac{[a]p, [b]p \vdash [a \cup b]p}{[a] \vdash [b]p \rightarrow [a \cup b]p} (\rightarrow R) \quad \frac{[a] \vdash [b]p \rightarrow [a \cup b]p}{[a]p \rightarrow ([b]p \rightarrow [a \cup b]p)} (\rightarrow R)
\]
A sequent calculus for PDL

Let CD be the following proof system where $g$ is $\emptyset$ or a singleton.

\[
\text{\textbf{(-R)}} \quad \frac{f, X \vdash \neg}{f, \vdash \neg X}
\]

\[
\text{\textbf{(-L)}} \quad \frac{f \vdash X}{f, \neg X \vdash}
\]

\[
\text{\textbf{\rightarrow R)}} \quad \frac{f, X \vdash Y}{f \vdash X \rightarrow Y}
\]

\[
\text{\textbf{\rightarrow L)}} \quad \frac{f \vdash X \quad f, Y \vdash g}{f, X \rightarrow Y \vdash g}
\]

\[
\text{\textbf{(; R)}} \quad \frac{f \vdash [\alpha][\beta]X}{f \vdash [\alpha; \beta]X}
\]

\[
\text{\textbf{(; L)}} \quad \frac{f, [\alpha][\beta]X \vdash g}{f, [\alpha; \beta]X \vdash g}
\]

\[
\text{\textbf{\cup R)}} \quad \frac{f \vdash [\alpha]X \quad f \vdash [\beta]X}{f \vdash [\alpha \cup \beta]X}
\]

\[
\text{\textbf{\cup L)}} \quad \frac{f, [\alpha]X, [\beta]X \vdash g}{f, [\alpha \cup \beta]X \vdash g}
\]

\[
\text{\textbf{(*L)}} \quad \frac{f, X, [\alpha][\alpha^*]X \vdash g}{f, [\alpha^*]X \vdash g}
\]

\[
\text{\textbf{(*R)}} \quad \frac{f \vdash \phi \quad f \vdash [\alpha]\phi \quad \cdots \quad f \vdash [\alpha]^k\phi}{f \vdash [\alpha^*]\phi}
\]

where $k = 2^{|f| + |\phi|}$

\[
\text{\textbf{(GEN)}} \quad \frac{f \vdash X}{[\alpha]f \vdash [\alpha]X}
\]

\[
\text{\textbf{(WEAK)}} \quad \frac{f \vdash g}{f' \vdash g'}
\]

where $f \subseteq f'$ and $g \subseteq g'$.
**Theorem** (Leivant 1981)
CD is a intuitionistic/constructive variant of D which is a sound and complete system for PDL, i.e. we have:

$$\models X \iff \vdash_D X \iff \vdash_{CD} X^0$$

where $X^0$ is the result of inserting $\neg\neg$ in front of everything in $X$.

NB: CD is *not* sound and complete for intuitionistic/constructive PDL.

Remaining goal: Show that CD has interpolation.
Maehara’s method

Idea
Find interpolants by going along the proof tree. Given the previous interpolants, we define the next one.

Example
Suppose the last step is $\cup R$:

\[
\begin{array}{c}
\vdash f \mid [\alpha]X \\
\vdash f \mid [\beta]X \\
\hline
\vdash f \mid [\alpha \cup \beta]X
\end{array}
\]

(\cup R)

Given any two interpolants $Z_1$ and $Z_2$ for $f \vdash [\alpha]X$ and $f \vdash [\beta]X$, let $Z := Z_1 \land Z_2 = \neg (Z_1 \rightarrow \neg Z_2)$. This interpolates $f \vdash [\alpha \cup \beta]X$. 
Definition
Given a sequent $f \vdash X$ and a partition of $f$ into $f^-; f^+$, we say that $K$ is an interpolant for $f^-; f^+ \vdash X$ iff

$$L(K) \subseteq L(f^-) \cap L(f^+, X) \text{ and } f^- \vdash K \text{ and } f^+, K \vdash X$$

Lemma 5.3.1 (Leivant 1981)
Let $f^-; f^+$ be any partition of $f$ and $q$ not occur in $f$.

(i) If $f \vdash_{CD} X$, then there is an interpolant for $f^-; f^+ \vdash X$.

(ii) Suppose $P$ is a proof of $f \vdash [\alpha]q$ from $\{f_i \vdash q\}_{i<k}$ and let $f_i^-; f_i^+$ be the partitions of $f_i$ induced by $f^-; f^+$ for all $i < k$. If $K_i$ is an interpolant for $f_i^-; f_i^+ \vdash X$ for all $i < k$, then there is an interpolant of the form $\bigwedge_i [\beta_i]K_i$ for $f^-; f^+ \vdash [\alpha]X$.

Proof. By tree-induction on $P$, simultaneously for (i) and (ii).
Suppose the last step is $\rightarrow L$:

$$\frac{f \vdash X \quad f, Y \vdash Z}{f, X \rightarrow Y \vdash Z}$$

($\rightarrow L$)

Case a) partition $f^-, X \rightarrow Y; f^+$. By induction hypothesis:

- $f^+; f^- \vdash X$ (Note: flipped!) yields $K_1$ such that
  $$L(K_1) \subseteq L(f^+) \cap L(f^-, X)$$
  and $f^+ \vdash K_1$ and $f^-, K_1 \vdash X$

- $f^-, Y; f^+ \vdash Z$ yields $K_2$ such that
  $$L(K_2) \subseteq L(f^-, Y) \cap L(f^+, Z)$$
  and $f^-, Y \vdash K_2$ and $f^+, K_2 \vdash Z$

Let $K := K_1 \rightarrow K_2$. This is interpolates $f^-, X \rightarrow Y; f^+ \vdash Z$.

Case b) partition $f^-; X \rightarrow Y, f^+$. Then $K := K_1 \wedge K_2$ works.
Suppose the last step of \( P \) is \((\ast R)\). For each \( h = 1 \leq M \) let \( P_h \) be the proof of \( f \vdash [\alpha]^hX \) occurring in \( P \) above this premise:

\[
\frac{P_0}{f \vdash X} \quad \frac{P_1}{f \vdash [\alpha]X} \quad \cdots \quad \frac{P_M}{f \vdash [\alpha]^M X} \quad (\ast R)
\]

Note: all active formulas on the right. Hence, only consider the given partition \( f^{-}, f^{+} \) without further manipulation.

Given: \( M \) many interpolants. Goal: find a formula \( K \) such that

\[
L(K) \subseteq L(f^{-}) \cap L(f^{+}, [\alpha^*]X) \text{ and } f^{-} \vdash K \text{ and } f^{+}, K \vdash [\alpha^*]X
\]

How?!
Definition
The positive closure of $f$, denoted by $\text{PC}(f)$, is the smallest set $g \supseteq f$ such that:

- If $(X \rightarrow Y) \in g$, then $Y \in g$.
- If $[\alpha]X \in g$, then $X \in g$.
- If $[\alpha; \beta]X \in g$, then $[\alpha][\beta]X \in g$.
- If $[\alpha \cup \beta]X \in g$, then $[\alpha]X \in g$ and $[\beta]X \in g$.
- If $[\alpha^*]X \in g$, then $[\alpha][\alpha^*]X \in g$.

Note: Whenever $f$ is finite, $\text{PC}(f)$ is also finite.
In certain proofs, $\text{PC}(\cdot)$ is preserved in the following sense.

**Lemma 4.2.1** (Leivant 1981, revision Venema 2014)

If $P$ proves $f \vdash [\beta_1] \ldots [\beta_k][\alpha]^m q$ from $\{f_i \vdash q\}_i$ where $q \notin L(f)$, all $\beta_i$s are subprograms of $\alpha$, $r < m$ and $f' \vdash [\alpha]^r q$ is a sequent in $P$ (under a non-initial leaf) then $\text{PC}(f') \subseteq \text{PC}(f)$.

The case we need is $k = 0$. 

Nice property 2

Definition
Let $P[X/q]$ be the result of substituting $X$ for $q$ in $P$.

Lemma 4.2.2 (Leivant 1981)
Suppose $P$ proves $f \vdash [\alpha]'X$ from $\{f_i \vdash X\}_i$ where $X \not\in PC(f)$. Then there is a proof $P'$ of $f \vdash [\alpha]'q$ from $\{f'_i \vdash q\}_i$ such that $P = P'[X/q]$.

Intuitively, this means that $P$ does not take $X$ apart:

$$
\frac{\{f_i \vdash X\}_i}{f \vdash [\alpha]'X} = \left( \frac{\{f_i \vdash q\}_i}{f \vdash [\alpha]'q} \right) [X/q]
$$
Nice property 3: Step by Step

Suppose $P$ is a CD-proof of $f \vdash [\alpha]^n X$. Then $P$ consists of proof parts $P_0, \ldots, P_n$ which build up the $[\alpha]$s “step by step”:

\[
\begin{align*}
\frac{P_0}{\{f_j \vdash X\}_{j \in I_0}} \\
\frac{P_1}{\{f_j \vdash [\alpha]X\}_{j \in I_1}} \\
\frac{P_2}{\{f_j \vdash [\alpha]^2X\}_{j \in I_2}} \\
\vdots \\
\frac{P_n}{f \vdash [\alpha]^n X}
\end{align*}
\]

NB: This looks more linear than it actually is!
Linear Transformations

Think of programs and formulas as a vector space:

$$(β) \bar{Y} = \begin{pmatrix} β_{1,1} & \cdots & β_{1,k} \\ \vdots & \ddots & \vdots \\ β_{k,1} & \cdots & β_{k,k} \end{pmatrix} \begin{pmatrix} Y_1 \\ \vdots \\ Y_k \end{pmatrix} := \begin{pmatrix} [β_{1,1}]Y_1 \wedge \cdots \wedge [β_{1,k}]Y_k \\ \vdots \\ [β_{k,1}]Y_1 \wedge \cdots \wedge [β_{k,k}]Y_k \end{pmatrix}$$

**Lemma**

For every $k \times k$ matrix $(β)$ there exists a $(γ)$ such that

$$(γ) \equiv (β)^* = (β)(β)(β) \cdots$$
Example

Let \( \vec{Y} = \langle p, q \rangle \) and \( (\beta) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Then \( (\beta) \vec{Y} = \begin{pmatrix} [a]p \land [b]q \\ [c]p \land [d]q \end{pmatrix} \)

and \( (\beta)(\beta) \vec{Y} = \begin{pmatrix} [a]([a]p \land [b]q) \land [b]([c]p \land [d]q) \\ [c]([a]p \land [b]q) \land [d]([c]p \land [d]q) \end{pmatrix} \).

Let \( \gamma := \begin{pmatrix} (a \cup (b; (d^*; c)))^* & (a^*; b)((c; a^*; b) \cup d)^* \\ (d^*; c)(a \cup (b; (d^*; c)))^* & ((c; a^*; b) \cup d)^* \end{pmatrix} \)

Then \( \gamma \equiv (\beta)^* \) and \( (\beta)^* \vec{Y} \equiv (\gamma) \vec{Y} \).

This \( \gamma \) can be found systematically. Moreover, it is useful:

\[
p \land [a]p \land ([a]p \land [b]q) \land ([a]([a]p \land [b]q) \land [b]([c]p \land [d]q)) \land \ldots
\]

\[
\equiv \quad [(a \cup (b; (d^*; c)))^*]p \land [(a^*; b)((c; a^*; b) \cup d)^*]q
\]
Putting it all together

Back to the evil * case

Now we can deal with this:

Now we can deal with this:

\[
\begin{align*}
P_0 & \quad f \vdash X \\
P_1 & \quad f \vdash [\alpha]X \\
\vdots & \\
P_M & \quad f \vdash [\alpha]^M X
\end{align*}
\]

\((*R)\)

Fix a ridiculously large \( h := s + v + d \) where

- \( d \) such that \([\alpha]^d X \not\in PC(f)\)
- \( v := 2^{|PC(f)|} \cdot 2^{|f|} + 1 \)
- \( s := 1 \) (for now).

Apply the step by step property to \( P_h \):

\[
\begin{align*}
Q_i \\
\{ f_i^-; f_i^+ \vdash [\alpha]^d X \}_{i \in I_d} \\
\vdots \\
f^-; f^+ \vdash [\alpha]^{d+v+s} X
\end{align*}
\]
Putting it all together
Finding a repetitive pattern

Now $P_h$ has to look like this:

\[
\begin{align*}
Q_{j,i} \\
\{f_i^-; f_i^+ \vdash [\alpha]^d X\}_{i \in I_d} \\
R'_j[[\alpha]^d X / q] \\
\{f_j^-; f_j^+ \vdash [\alpha]^{d+v} X\}_{j \in I_{d+v}} \\
U'[[\alpha]^{d+v} X / q] \\
f^-; f^+ \vdash [\alpha]^{d+v+s} X
\end{align*}
\]

For all $c \leq v$, $j \in I_{d+c}$: $f_i \subseteq PC(f_i) \subseteq PC(f)$ and $|\mathcal{P}(f_j)| \leq |\mathcal{P}(f)|$

Hence $|\bigcup \{\mathcal{P}(f_j) \mid c \leq v, j \in I_{d+c}\} \leq |\mathcal{P}(PC(f))| \cdot |\mathcal{P}(f)|$

$= 2|PC(f)| \cdot 2^{|f|} = v - 1 < v$.

**Repetitive Pattern**

For some $m \neq n$ we have \{\(f_j^+; f_j^- \mid j \in I_m\} = \{f_j^+; f_j^- \mid j \in I_n\}$.

Furthermore, we can assume $d < m < n < d + v$ and $I_m = I_n$. 
Let $r$ be such that $n = m + r$. Now $P_h$ can be divided as follows:

\[
\begin{align*}
Q_i & \quad \frac{\{f_i^-; f_i^+ \vdash [\alpha]^m X\}_{i \in I}}{R'_j[[\alpha]^m X/q]} \\
& \quad \frac{\{f_j^-; f_j^+ \vdash [\alpha]^{m+r} X\}_{j \in I}}{U'[\alpha]^{m+r} X/q} \\
& \quad \frac{f_i^-; f_i^+ \vdash [\alpha]^{m+r+s} X}{\text{IH}(i) \text{ yields } \vec{K} \text{ such that } K_i \text{ interpolates } f_i^-; f_i^+ \vdash [\alpha]^m X. \text{ Using } \\
\text{IH}(ii) \text{ } r \text{ times: If } \vec{M} \text{ contains interpolants for } f_i^-; f_i^+ \vdash Y \text{ then there is a matrix } (\beta) \text{ such that } ((\beta)M)_i \text{ interpolates } f_i^-; f_i^+ \vdash [\alpha]^r Y.}
\end{align*}
\]

Thus, for all $n$, by applying the latter to the former $n$ times:

\[
f_i^- \vdash ((\beta)^n K)_i \quad \text{and} \quad f_i^+, ((\beta)^n K)_i \vdash [\alpha]^m [\alpha]^{r \times n} X
\]
Putting it all together

Done, repeat.

By linear transformations there is a $\gamma$ such that:

$$f_i^- \vdash ((\gamma)K)_i \quad \text{and} \quad f_i^+ , (\gamma)K)_i \vdash [\alpha]^m[(\alpha^r)^*]X$$

Now apply IH(ii) to all the $((\gamma)K)_i$s and $U'$.
This yields an interpolant $H_s$ for $f^- ; f^+ \vdash [\alpha]^s[\alpha]^m[(\alpha^r)^*]X$.
Repeat all of the above to obtain $H_1, \ldots, H_{v+d}$.

Finally, let $K := \bigwedge_{s \leq v+d} H_s$. This interpolates $f^- ; f^+ \vdash [\alpha^*]X$.

**Lemma**

$$\vdash_{\text{CD}} \bigwedge_{k \leq w} [\alpha^k][(\alpha^w)^*]X \rightarrow [\alpha^*]X.$$
Theorem 5.3.2 (i) (Leivant 1981)
PDL has Craig Interpolation.

Proof. Take any $\models X \rightarrow Y$. $D$ is complete, hence $\vdash_D X \rightarrow Y$.
Then $\vdash_{CD} X^o \rightarrow Y^o$ and thus $X^o \vdash_{CD} Y^o$.
Partition-interpolation of $X^o; \emptyset \vdash Y^o$ yields $Z$ such that

- $L(Z) \subseteq L(X^o) \cap L(\emptyset, Y^o)$,
- $X^o \rightarrow Z \in \text{PDL}$ and $Z \rightarrow Y^o \in \text{PDL}$

By $X^o \equiv X$, $Y^o \equiv Y$, $L(X^o) = L(X)$ and $L(Y^o) = L(Y)$:

- $L(Z) \subseteq L(X) \cap L(Y)$,
- $X \rightarrow Z \in \text{PDL}$ and $Z \rightarrow Y \in \text{PDL}$

Hence $Z$ is an interpolant for $X \rightarrow Y$. □
Criticism
Criticism


“[T]he problem of interpolation for PDL is one of the major open problems in this area. Twice a solution has been announced [...], but in neither case was it possible to verify the argument. The argument of Leivant makes use of the fact that if $\phi \vdash_{PDL} \psi$ then we can bound the size of a possible countermodel so that the star $\alpha^*$ only needs to search up to a depth $d$ which depends on $\phi$ and $\psi$.”[8, p. 493]
Marcus Kracht (continued):

“The argument of Leivant makes use of the fact that if \( \phi \vdash_{PDL} \psi \) then we can bound the size of a possible countermodel so that the star \( \alpha^* \) only needs to search up to a depth \( d \) which depends on \( \phi \) and \( \psi \). Once that is done, we have reduced \( PDL \) to \( EPDL \), which definitely has interpolation because it is a notational variant of polymodal \( K \). However, this is tantamount to the following. Abbreviate by \( PDL^n \) the strengthening of \( PDL \) by axioms of the form \([a^*]p \leftrightarrow [a^{\leq n}]p\) for all \( a \). Then, by the finite model property of \( PDL \), \( PDL \) is the intersection of the logics \( PDL^n \). Unfortunately, it is not so that interpolation is preserved under intersection.”[8, p. 493]
PDL and $\text{PDL}^n$

**Definition**

Semantic closure $\text{SCL}(A) := \{\phi \mid A \vDash \phi\}$

$[\alpha \leq^n \phi] := \phi \land [\alpha] \phi \land [\alpha; \alpha] \phi \land \cdots \land [\alpha^n] \phi$

$\text{PDL}^n:= \text{SCL}(\text{PDL} \cup \{[\alpha^*]p \leftrightarrow [\alpha \leq^n]p \mid \alpha \in \text{PROG}, p \in \mathcal{P}\})$

**Theorem**

$\text{PDL}^0 \supseteq \text{PDL}^1 \supseteq \text{PDL}^2 \supseteq \cdots \supseteq \text{PDL} = \bigcap_n \text{PDL}^n$

**Idea / Question**

Is there an $n$, depending on $|\phi \to \psi|$ such that any $\text{PDL}^n$-interpolant for $\phi \to \psi$ is also a $\text{PDL}$-interpolant?
But this is not what Leivant is doing:

\[ f \vdash \phi \quad f \vdash [\alpha] \phi \quad \cdots \quad f \vdash [\alpha]^k \phi \]

\[ f \vdash [\alpha^*] \phi \]

where \( k = 2^{|f|+|\phi|} \) and therefore depends on \( f \) and \( \phi \).

**Theorem: Finite-Model Property**
If \( \phi \) is satisfiable, then there is a model \( \mathcal{M} = (W, R, V) \) and a world \( w \in W \) such that \( \mathcal{M}, w \models \phi \) and \( |W| \leq 2^{\text{size}(\phi)} \).

**Lemma**
If \( \models \bigwedge f \rightarrow [\alpha]^n \phi \) for all \( n \leq k = 2^{|f|+|\phi|} \), then \( \models \bigwedge f \rightarrow [\alpha^*] \phi \).

**Theorem**
The finitary rule is admissible.
Conclusion
Conclusion

- There is a finitary sequent calculus for PDL. (In particular, Kracht’s criticism does not apply.)
- This system has the “step by step” property.
- Therefore we can:
  - find a repetitive pattern in long enough proofs.
  - use linear transformations to build * interpolants.
- This extends Maehara’s method to show Craig Interpolation.

All this [c u sh]ould have been known since 1981.

Moreover, can this proof also be done in multi-type calculi?
Epilogue

Kracht: “Twice a solution has been announced ...”
Borzechowski 1988: unpublished, unknown and unread?
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Thank you!

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